

Radicals and surds

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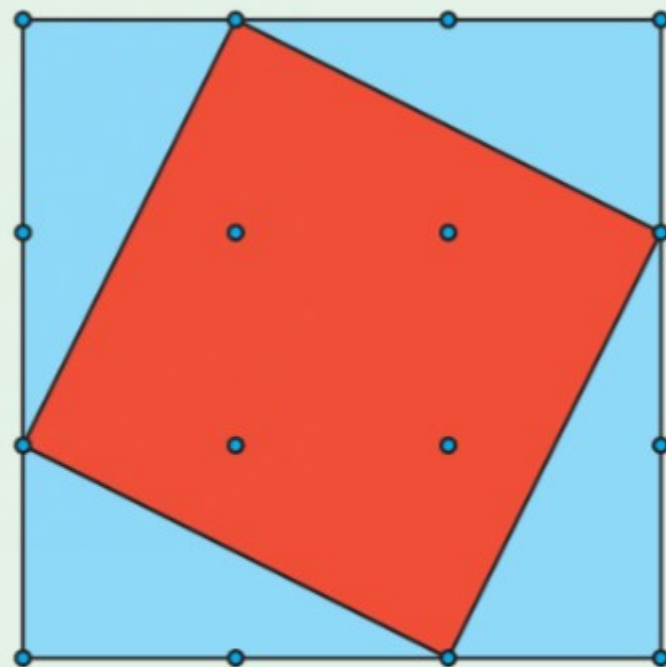
OPENING PROBLEM

Pamela's students claim that since an irrational number cannot be written as a fraction, it cannot be placed on a number line.

To demonstrate that an irrational length can be represented, and therefore placed on a number line, Pamela draws this figure.

Things to think about:

- a** Find:
 - i** the area of the large square
 - ii** the total area of the blue triangles.
- b** Using your answers to **a**, find the area of the red square.
- c** Hence explain why the side length of the red square is $\sqrt{5}$ units.
- d** What is the perimeter of the red square?



In **Chapter 1** we encountered values such as $\sqrt{3}$, $\sqrt{5}$, and $\sqrt[3]{8}$. These numbers are known as *radicals*.

A **radical** is a number that is written using the radical sign $\sqrt{}$.

Radicals occur frequently in mathematics, often as solutions to equations involving squared terms. We will see a typical example of this in **Chapter 5** when we study Pythagoras' theorem.

SQUARE ROOTS

The **square root of a** , written $\sqrt{a}$, is the *positive* solution of the equation $x^2 = a$.
 $\sqrt{a}$ is the *positive* number which obeys the rule $\sqrt{a} \times \sqrt{a} = a$.

For example, $\sqrt{2} \times \sqrt{2} = 2$, $\sqrt{3} \times \sqrt{3} = 3$, $\sqrt{4} \times \sqrt{4} = 4$, and so on.

We know that the square of any real number is non-negative. This means that:

$\sqrt{a}$ is real only if $a \geq 0$.

HIGHER ROOTS

In this course we will concentrate mainly on square roots, but it is important to understand that other radicals exist. For example:

The **cube root of** a , written $\sqrt[3]{a}$, satisfies the rule $(\sqrt[3]{a})^3 = a$.

If $a > 0$ then $\sqrt[3]{a} > 0$.

If $a < 0$ then $\sqrt[3]{a} < 0$.

We can define higher roots in a similar way.

RATIONAL AND IRRATIONAL RADICALS

In **Chapter 2** we saw that the set of **real numbers** $\mathbb{R}$ can be divided into the set of **rational numbers** $\mathbb{Q}$, and the set of **irrational numbers** $\mathbb{Q}'$.

Remember that:

An **irrational number** is a real number which cannot be written in the form $\frac{p}{q}$, where p and q are integers, $q \neq 0$.

Radical numbers may be rational or irrational. An irrational radical is called a surd.

Examples of rational radicals include:

$$\sqrt{4} = 2 = \frac{2}{1}$$

$$\sqrt{\frac{9}{16}} = \sqrt{\left(\frac{3}{4}\right)^2} = \frac{3}{4}$$

Two examples of surds are $\sqrt{2} \approx 1.414\,214$
and $\sqrt{3} \approx 1.732\,051$.

If the number under the radical sign
can be written as a perfect square,
then the radical is rational.



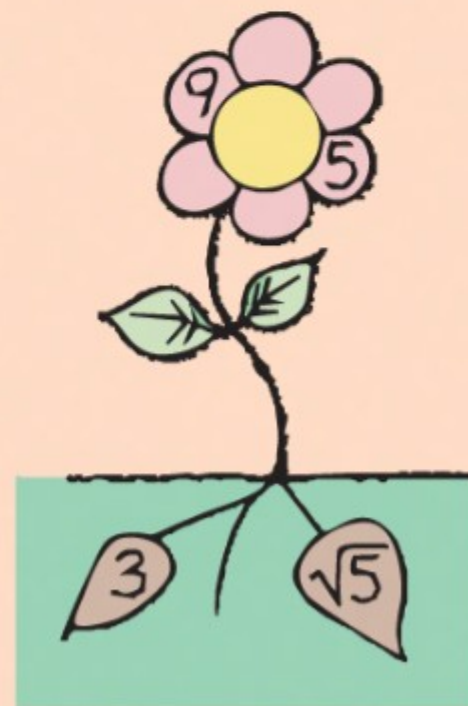
HISTORICAL NOTE

When the Golden Age of the Greeks was past, the writings of the Greeks were preserved, translated into Arabic and extended by Arabic mathematicians in the regions currently known as Iraq and Iran and also in Moslem Spain. The word *surd* came about because of an error in translation.

The Greek word *alogos* meaning ‘irrational’, or ‘without reason’, was translated as the Arabic word *asamm* which means ‘irrational’, but also means ‘deaf’. Thus rational and irrational numbers were called ‘audible’ and ‘inaudible’ numbers respectively by the Arabic mathematician **Al-Khwarizmi**, from Persia, around 825 AD.

This later led to the Arabic word *asamm* meaning ‘deaf’ or ‘dumb’ for irrational numbers being translated into Latin as *surdus* meaning ‘deaf’ or ‘mute’. The European mathematician, Gherardo of Cremona (c. 1150), adopted the word *surd*.

The origin of the root symbol $\sqrt{\quad}$ is not clear. Some sources suggest that the symbol was first used by Arabic mathematicians. It is believed that the modern square root symbol developed from the letter *r* which is the first letter in the Latin word *radix*, meaning ‘root’, from which we get the word *radical*.



SIMPLIFYING RADICALS

In previous years we have established that:

- $\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$ for $a \geq 0, b \geq 0$
- $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$ for $a \geq 0, b > 0$

Example 1

 Self Tutor

Simplify:

a $(\sqrt{5})^2$

b $\left(\frac{1}{\sqrt{5}}\right)^2$

Example 1



Simplify:

a $(\sqrt{5})^2$

b $\left(\frac{1}{\sqrt{5}}\right)^2$

a

$$\begin{aligned} & (\sqrt{5})^2 \\ &= \sqrt{5} \times \sqrt{5} \\ &= 5 \end{aligned}$$

b

$$\begin{aligned} & \left(\frac{1}{\sqrt{5}}\right)^2 \\ &= \frac{1}{\sqrt{5}} \times \frac{1}{\sqrt{5}} \\ &= \frac{1}{5} \end{aligned}$$

Example 2



Simplify:

a $(2\sqrt{5})^3$

b $-2\sqrt{5} \times 3\sqrt{5}$

Example 2

Simplify:

a $(2\sqrt{5})^3$

b $-2\sqrt{5} \times 3\sqrt{5}$

a

$$\begin{aligned}(2\sqrt{5})^3 &= 2\sqrt{5} \times 2\sqrt{5} \times 2\sqrt{5} \\ &= 2 \times 2 \times 2 \times \sqrt{5} \times \sqrt{5} \times \sqrt{5} \\ &= 8 \times 5 \times \sqrt{5} \\ &= 40\sqrt{5}\end{aligned}$$

b

$$\begin{aligned}-2\sqrt{5} \times 3\sqrt{5} &= -2 \times 3 \times \sqrt{5} \times \sqrt{5} \\ &= -6 \times 5 \\ &= -30\end{aligned}$$

1 Simplify:

a $(\sqrt{7})^2$

b $(\sqrt{13})^2$

c $(\sqrt{15})^2$

d $(\sqrt{24})^2$

e $(\sqrt{2})^3$

f $(\sqrt{2})^4$

g $(\sqrt{2})^5$

h $\left(\frac{1}{\sqrt{2}}\right)^2$

i $\left(\frac{1}{\sqrt{3}}\right)^2$

j $\left(\frac{2}{\sqrt{11}}\right)^2$

k $\left(\frac{\sqrt{3}}{\sqrt{17}}\right)^2$

l $\left(\sqrt{\frac{2}{23}}\right)^2$

2 Simplify:

a $(\sqrt[3]{2})^3$

b $(\sqrt[3]{-5})^3$

c $\left(\frac{1}{\sqrt[3]{5}}\right)^3$

3 Simplify:

a $4\sqrt{3} \times \sqrt{3}$

b $3\sqrt{2} \times \sqrt{2}$

c $\sqrt{5} \times 6\sqrt{5}$

d $3\sqrt{2} \times 4\sqrt{2}$

e $-2\sqrt{3} \times 5\sqrt{3}$

f $3\sqrt{5} \times (-2\sqrt{5})$

g $-2\sqrt{2} \times (-3\sqrt{2})$

h $(3\sqrt{2})^2$

i $(3\sqrt{2})^3$

j $(4\sqrt{3})^2$

k $(4\sqrt{3})^3$

l $(2\sqrt{2})^4$

m $\sqrt{5} \times (3\sqrt{5})^2$

n $-2\sqrt{3} \times (5\sqrt{3})^2$

o $(2\sqrt{2})^3 \times (-7\sqrt{2})$

Example 3

Self Tutor

Write in simplest form:

a $\sqrt{3} \times \sqrt{2}$

b $2\sqrt{5} \times 3\sqrt{2}$

With practice you should not need the middle steps.



Example 3

Self Tutor

Write in simplest form:

a $\sqrt{3} \times \sqrt{2}$

b $2\sqrt{5} \times 3\sqrt{2}$

a

$$\begin{aligned}\sqrt{3} \times \sqrt{2} \\&= \sqrt{3 \times 2} \\&= \sqrt{6}\end{aligned}$$

b

$$\begin{aligned}2\sqrt{5} \times 3\sqrt{2} \\&= 2 \times 3 \times \sqrt{5} \times \sqrt{2} \\&= 6 \times \sqrt{5 \times 2} \\&= 6\sqrt{10}\end{aligned}$$

With practice you should not need the middle steps.



4 Simplify:

a $\sqrt{2} \times \sqrt{5}$

d $\sqrt{7} \times \sqrt{7}$

g $3\sqrt{3} \times 2\sqrt{2}$

j $\sqrt{3} \times \sqrt{2} \times 2\sqrt{2}$

b $\sqrt{3} \times \sqrt{7}$

e $\sqrt{3} \times 2\sqrt{3}$

h $2\sqrt{3} \times 3\sqrt{5}$

k $-3\sqrt{2} \times (\sqrt{2})^3$

c $\sqrt{3} \times \sqrt{11}$

f $2\sqrt{2} \times (-\sqrt{5})$

i $\sqrt{2} \times \sqrt{3} \times \sqrt{5}$

l $(3\sqrt{2})^3 \times (\sqrt{3})^3$

Example 4 **Self Tutor**

Simplify:

a $\frac{\sqrt{32}}{\sqrt{2}}$

b $\frac{\sqrt{12}}{2\sqrt{3}}$

Example 4 **Self Tutor**

Simplify:

a $\frac{\sqrt{32}}{\sqrt{2}}$

b $\frac{\sqrt{12}}{2\sqrt{3}}$

a
$$\begin{aligned}\frac{\sqrt{32}}{\sqrt{2}} &= \sqrt{\frac{32}{2}} \\ &= \sqrt{16} \\ &= 4\end{aligned}$$

b
$$\begin{aligned}\frac{\sqrt{12}}{2\sqrt{3}} &= \frac{1}{2}\sqrt{\frac{12}{3}} \quad \left\{ \text{using } \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}} \right\} \\ &= \frac{1}{2}\sqrt{4} \\ &= \frac{1}{2} \times 2 \\ &= 1\end{aligned}$$

5 Simplify:

a $\frac{\sqrt{8}}{\sqrt{2}}$

b $\frac{\sqrt{2}}{\sqrt{8}}$

c $\frac{\sqrt{18}}{\sqrt{2}}$

d $\frac{\sqrt{2}}{\sqrt{18}}$

e $\frac{\sqrt{20}}{\sqrt{5}}$

f $\frac{\sqrt{5}}{\sqrt{20}}$

g $\frac{\sqrt{27}}{\sqrt{3}}$

h $\frac{\sqrt{18}}{\sqrt{3}}$

i $\frac{\sqrt{3}}{\sqrt{30}}$

j $\frac{\sqrt{50}}{\sqrt{2}}$

k $\frac{2\sqrt{6}}{\sqrt{24}}$

l $\frac{5\sqrt{75}}{\sqrt{3}}$

6 Simplify:

a $\sqrt{\frac{1}{25}}$

b $\sqrt{\frac{16}{9}}$

c $\sqrt{3\frac{1}{16}}$

d $\sqrt{20\frac{1}{4}}$

7 a Prove that $\sqrt{a}\sqrt{b} = \sqrt{ab}$ for all positive numbers a and b .

Hint: Consider $(\sqrt{a}\sqrt{b})^2$ and $(\sqrt{ab})^2$.

b Prove that $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$ for $a \geq 0$ and $b > 0$.

- 8** **a** Is $\sqrt{9} + \sqrt{16} = \sqrt{9 + 16}$? Is $\sqrt{25} - \sqrt{16} = \sqrt{25 - 16}$?
- b** Are $\sqrt{a} + \sqrt{b} = \sqrt{a + b}$ and $\sqrt{a} - \sqrt{b} = \sqrt{a - b}$ possible laws for radicals?

$$\begin{array}{llllll} \mathbf{1} & \mathbf{a} & 7 & \mathbf{b} & 13 & \mathbf{c} & 15 & \mathbf{d} & 24 & \mathbf{e} & 2\sqrt{2} & \mathbf{f} & 4 \\ & \mathbf{g} & 4\sqrt{2} & \mathbf{h} & \frac{1}{2} & \mathbf{i} & \frac{1}{3} & \mathbf{j} & \frac{4}{11} & \mathbf{k} & \frac{3}{17} & \mathbf{l} & \frac{2}{23} \end{array}$$

$$\mathbf{2} \quad \mathbf{a} \quad 2 \quad \mathbf{b} \quad -5 \quad \mathbf{c} \quad \frac{1}{5}$$

$$\begin{array}{llllll} \mathbf{3} & \mathbf{a} & 12 & \mathbf{b} & 6 & \mathbf{c} & 30 & \mathbf{d} & 24 & \mathbf{e} & -30 \\ & \mathbf{f} & -30 & \mathbf{g} & 12 & \mathbf{h} & 18 & \mathbf{i} & 54\sqrt{2} & \mathbf{j} & 48 \\ & \mathbf{k} & 192\sqrt{3} & \mathbf{l} & 64 & \mathbf{m} & 45\sqrt{5} & \mathbf{n} & -150\sqrt{3} & \mathbf{o} & -224 \end{array}$$

$$\begin{array}{llll} \mathbf{4} & \mathbf{a} & \sqrt{10} & \mathbf{b} & \sqrt{21} & \mathbf{c} & \sqrt{33} & \mathbf{d} & 7 \\ & \mathbf{e} & 6 & \mathbf{f} & -2\sqrt{10} & \mathbf{g} & 6\sqrt{6} & \mathbf{h} & 6\sqrt{15} \\ & \mathbf{i} & \sqrt{30} & \mathbf{j} & 4\sqrt{3} & \mathbf{k} & -12 & \mathbf{l} & 162\sqrt{6} \end{array}$$

$$\begin{array}{llllll} \mathbf{5} & \mathbf{a} & 2 & \mathbf{b} & \frac{1}{2} & \mathbf{c} & 3 & \mathbf{d} & \frac{1}{3} & \mathbf{e} & 2 & \mathbf{f} & \frac{1}{2} \\ & \mathbf{g} & 3 & \mathbf{h} & \sqrt{6} & \mathbf{i} & \frac{1}{\sqrt{10}} & \mathbf{j} & 5 & \mathbf{k} & 1 & \mathbf{l} & 25 \end{array}$$

$$\mathbf{6} \quad \mathbf{a} \quad \frac{1}{5} \quad \mathbf{b} \quad \frac{4}{3} \quad \mathbf{c} \quad \frac{7}{4} \quad \mathbf{d} \quad \frac{9}{2}$$

$$\mathbf{7} \quad \mathbf{a} \quad (\sqrt{a}\sqrt{b})^2 = \sqrt{a}\sqrt{b} \times \sqrt{a}\sqrt{b} = ab$$

$$(\sqrt{ab})^2 = \sqrt{ab} \times \sqrt{ab} = ab$$

$$\text{As } a \geq 0, b \geq 0, (\sqrt{a}\sqrt{b})^2 = (\sqrt{ab})^2$$

$$\mathbf{b} \quad \text{Using } \mathbf{a}, \quad \sqrt{b} \times \sqrt{\frac{a}{b}} = \sqrt{b \times \frac{a}{b}} = \sqrt{a}$$

$$\therefore \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

$$\mathbf{8} \quad \mathbf{a} \quad \sqrt{9} + \sqrt{16} = 3 + 4 = 7, \text{ whereas } \sqrt{9+16} = \sqrt{25} = 5$$

$$\therefore \sqrt{9} + \sqrt{16} \neq \sqrt{9+16}$$

$$\sqrt{25} - \sqrt{16} = 5 - 4 = 1, \text{ whereas } \sqrt{25-16} = \sqrt{9} = 3$$

$$\therefore \sqrt{25} - \sqrt{16} \neq \sqrt{25-16}$$

$$\mathbf{b} \quad \text{No, as they are not true for all } a \geq 0, b \geq 0.$$